5.6 Inverse Trigonometric Functions: Differentiation

- **Develop properties of the six inverse trigonometric functions.**
- **Differentiate an inverse trigonometric function.**
- **Review the basic differentiation rules for elementary functions.**

Inverse Trigonometric Functions

This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section, you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains.*

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval

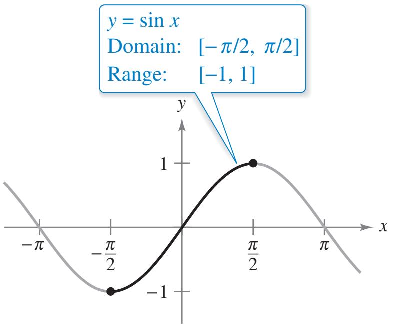
$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

as shown in Figure 5.25. On this interval, you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \leq x \leq 1$ and $-\pi/2 \leq \arcsin x \leq \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the next definition. (Note that the term "iff" is used to represent the phrase "if and only if.")



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.25

REMARK The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$."

Definitions of Inverse Trigonometric Functions

Function	**Domain**	**Range**
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot } x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec } x$ iff $\sec y = x$	$\lvert x \rvert \geq 1$	$0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}$
$y = \text{arccsc } x$ iff $\csc y = x$	$\lvert x \rvert \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0$

Exploration

The Inverse Secant Function In the definitions of the inverse trigonometric functions, the inverse secant function is defined by restricting the domain of the secant function to the intervals $[0, \pi/2) \cup (\pi/2, \pi]$. Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

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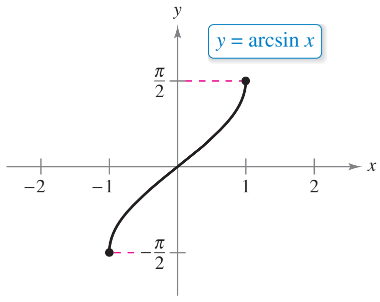
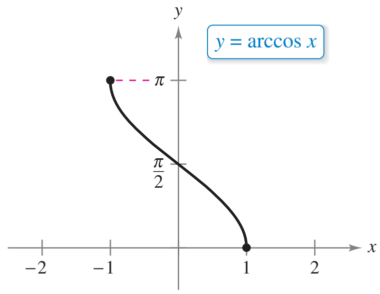
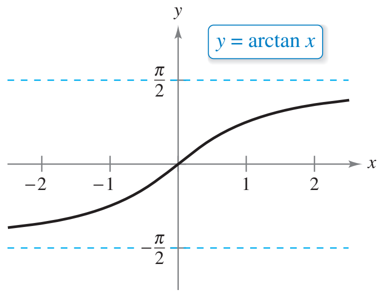
The graphs of the six inverse trigonometric functions are shown in Figure 5.26.



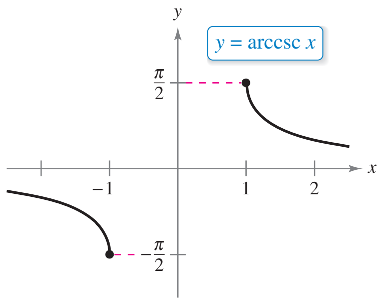
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



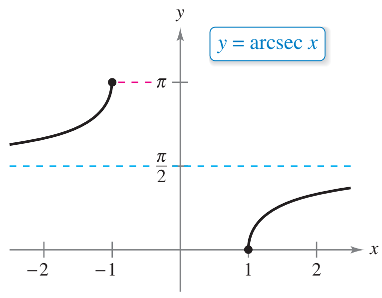
Domain: $[-1, 1]$
Range: $[0. \pi]$



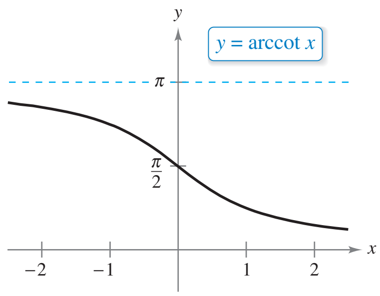
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Figure 5.26

When evaluating inverse trigonometric functions, remember that they denote angles in *radian measure*.

EXAMPLE 1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

a. $\arcsin\left(-\dfrac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.305.$$

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Inverse functions have the properties $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 2 Solving an Equation

$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Original equation}$$

$$\tan[\arctan(2x - 3)] = \tan \frac{\pi}{4} \qquad \text{Take tangent of each side.}$$

$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 3.

EXAMPLE 3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

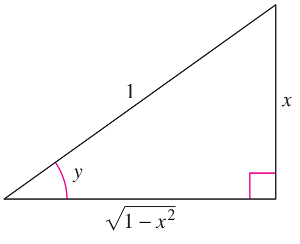
b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in the figure at the right.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

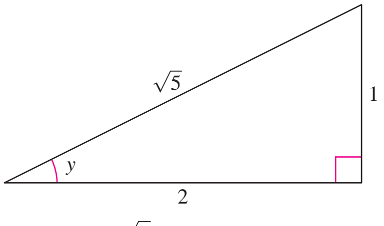
(This result is also valid for $-\pi/2 < y < 0$.)



$y = \arcsin x$

b. Use the right triangle shown in the figure at the left.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$



$y = \operatorname{arcsec} \frac{\sqrt{5}}{2}$

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Derivatives of Inverse Trigonometric Functions

In Section 5.1, you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The next theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

REMARK There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\operatorname{arcsec} x = \arccos \frac{1}{x}.$$

One consequence of this definition is that its graph has a positive slope at every x-value in its domain. (See Figure 5.26.) This accounts for the absolute value sign in the formula for the derivative of arcsec x.

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Proofs for arcsin u and arccos u are given in Appendix A. [The proofs for the other rules are left as an exercise (see Exercise 98).]

See LarsonCalculus.com for Bruce Edwards's video of this proof.

TECHNOLOGY If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \operatorname{arcsec} x = \arccos \frac{1}{x}.$$

EXAMPLE 4 Differentiating Inverse Trigonometric Functions

a. $\dfrac{d}{dx}[\arcsin(2x)] = \dfrac{2}{\sqrt{1-(2x)^2}} = \dfrac{2}{\sqrt{1-4x^2}}$

b. $\dfrac{d}{dx}[\arctan(3x)] = \dfrac{3}{1+(3x)^2} = \dfrac{3}{1+9x^2}$

c. $\dfrac{d}{dx}[\arcsin\sqrt{x}] = \dfrac{(1/2)x^{-1/2}}{\sqrt{1-x}} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{1}{2\sqrt{x-x^2}}$

d. $\dfrac{d}{dx}[\operatorname{arcsec} e^{2x}] = \dfrac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \dfrac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \dfrac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$.

EXAMPLE 5 A Derivative That Can Be Simplified

$$\begin{aligned}
y &= \arcsin x + x\sqrt{1-x^2} \\
y' &= \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2} \\
&= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \\
&= \sqrt{1-x^2} + \sqrt{1-x^2} \\
&= 2\sqrt{1-x^2}
\end{aligned}$$

From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right).$$

FOR FURTHER INFORMATION For more on the derivative of the arctangent function, see the article "Differentiating the Arctangent Directly" by Eric Key in *The College Mathematics Journal*. To view this article, go to *MathArticles.com.*

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EXAMPLE 6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$y' = 2(\arctan x)\left(\frac{1}{1+x^2}\right) = \frac{2\arctan x}{1+x^2}$$

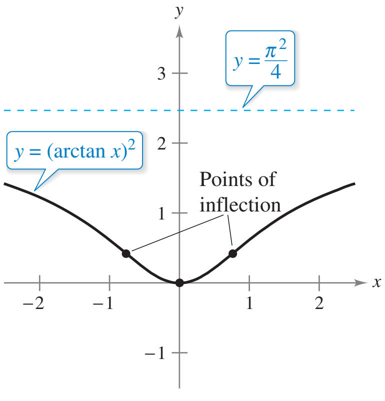
you can see that the only critical number is $x = 0$. By the First Derivative Test, this value corresponds to a relative minimum. From the second derivative

$$y'' = \frac{(1+x^2)\left(\frac{2}{1+x^2}\right) - (2\arctan x)(2x)}{(1+x^2)^2} = \frac{2(1-2x\arctan x)}{(1+x^2)^2}$$

it follows that points of inflection occur when $2x \arctan x = 1$. Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x\to\pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 5.27.

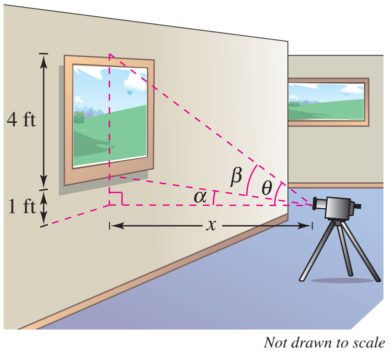


The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.27

EXAMPLE 7 Maximizing an Angle

See LarsonCalculus.com for an interactive version of this type of example.

A photographer is taking a picture of a painting hung in an art gallery. The height of the painting is 4 feet. The camera lens is 1 foot below the lower edge of the painting, as shown in the figure at the right. How far should the camera be from the painting to maximize the angle subtended by the camera lens?



The camera should be 2.236 feet from the painting to maximize the angle β.

Solution In the figure, let β be the angle to be maximized.

$$\beta = \theta - \alpha = \operatorname{arccot}\frac{x}{5} - \operatorname{arccot} x$$

Differentiating produces

$$\frac{d\beta}{dx} = \frac{-1/5}{1+(x^2/25)} - \frac{-1}{1+x^2} = \frac{-5}{25+x^2} + \frac{1}{1+x^2} = \frac{4(5-x^2)}{(25+x^2)(1+x^2)}.$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative Test that this distance yields a maximum value of β. So, the distance is $x \approx 2.236$ feet and the angle is $\beta \approx 0.7297$ radian $\approx 41.81°$.

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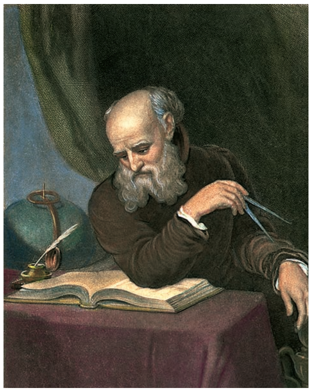

GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities*, such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities*, such as time, distance, force, and mass.

See LarsonCalculus.com to read more of this biography.

Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions have proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	**Transcendental Functions**
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\dfrac{d}{dx}[cu] = cu'$
2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$
3. $\dfrac{d}{dx}[uv] = uv' + vu'$
4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$
5. $\dfrac{d}{dx}[c] = 0$
6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \quad u \neq 0$
9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u u'$
11. $\dfrac{d}{dx}[\log_a u] = \dfrac{u'}{(\ln a)u}$
12. $\dfrac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\dfrac{d}{dx}[\sin u] = (\cos u)u'$
14. $\dfrac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\dfrac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$
20. $\dfrac{d}{dx}[\arccos u] = \dfrac{-u'}{\sqrt{1 - u^2}}$
21. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$
22. $\dfrac{d}{dx}[\text{arccot } u] = \dfrac{-u'}{1 + u^2}$
23. $\dfrac{d}{dx}[\text{arcsec } u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\dfrac{d}{dx}[\text{arccsc } u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$

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* Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.

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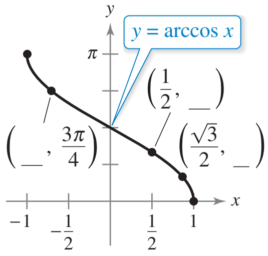
5.6 Exercises

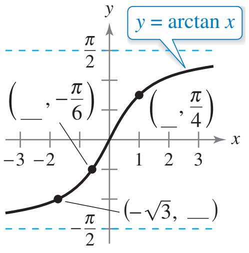
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Coordinates In Exercises 1 and 2, determine the missing coordinates of the points on the graph of the function.

1.



2.



Evaluating Inverse Trigonometric Functions In Exercises 3–10, evaluate the expression without using a calculator.

3. $\arcsin \frac{1}{2}$

4. $\arcsin 0$

5. $\arccos \frac{1}{2}$

6. $\arccos 1$

7. $\arctan \dfrac{\sqrt{3}}{3}$

8. $\operatorname{arccot}\left(-\sqrt{3}\right)$

9. $\operatorname{arccsc}\left(-\sqrt{2}\right)$

10. $\operatorname{arcsec}\left(-\sqrt{2}\right)$

Approximating Inverse Trigonometric Functions In Exercises 11–14, use a calculator to approximate the value. Round your answer to two decimal places.

11. $\arccos(-0.8)$

12. $\arcsin(-0.39)$

13. $\operatorname{arcsec} 1.269$

14. $\arctan(-5)$

Using a Right Triangle In Exercises 15–20, use the figure to write the expression in algebraic form given $y = \arccos x$, where $0 < y < \pi/2$.

15. $\cos y$

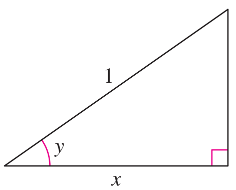
16. $\sin y$

17. $\tan y$

18. $\cot y$

19. $\sec y$

20. $\csc y$



Evaluating an Expression In Exercises 21–24, evaluate each expression without using a calculator. (*Hint:* See Example 3.)

21. (a) $\sin\left(\arctan \dfrac{3}{4}\right)$

(b) $\sec\left(\arcsin \dfrac{4}{5}\right)$

22. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right)$

(b) $\cos\left(\arcsin \dfrac{5}{13}\right)$

23. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right]$

(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right]$

24. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right]$

(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right]$

Simplifying an Expression Using a Right Triangle In Exercises 25–32, write the expression in algebraic form. (*Hint:* Sketch a right triangle, as demonstrated in Example 3.)

25. $\cos(\arcsin 2x)$

26. $\sec(\arctan 4x)$

27. $\sin(\operatorname{arcsec} x)$

28. $\cos(\operatorname{arccot} x)$

29. $\tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

30. $\sec[\arcsin(x - 1)]$

31. $\csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$

32. $\cos\left(\arcsin \dfrac{x - h}{r}\right)$

Solving an Equation In Exercises 33–36, solve the equation for x.

33. $\arcsin(3x - \pi) = \frac{1}{2}$

34. $\arctan(2x - 5) = -1$

35. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

36. $\arccos x = \operatorname{arcsec} x$

Verifying Identities In Exercises 37 and 38, verify each identity.

37. (a) $\operatorname{arccsc} x = \arcsin \dfrac{1}{x}, \quad x \geq 1$

(b) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

38. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

Finding a Derivative In Exercises 39–58, find the derivative of the function.

39. $f(x) = 2\arcsin(x - 1)$

40. $f(t) = \arcsin t^2$

41. $g(x) = 3\arccos \dfrac{x}{2}$

42. $f(x) = \operatorname{arcsec} 2x$

43. $f(x) = \arctan e^x$

44. $f(x) = \arctan\sqrt{x}$

45. $g(x) = \dfrac{\arcsin 3x}{x}$

46. $h(x) = x^2 \arctan 5x$

47. $h(t) = \sin(\arccos t)$

48. $f(x) = \arcsin x + \arccos x$

49. $y = 2x \arccos x - 2\sqrt{1 - x^2}$

50. $y = \ln(t^2 + 4) - \dfrac{1}{2}\arctan \dfrac{t}{2}$

51. $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln \dfrac{x + 1}{x - 1} + \arctan x\right)$

52. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4\arcsin\left(\dfrac{x}{2}\right)\right]$

53. $y = x\arcsin x + \sqrt{1 - x^2}$

54. $y = x\arctan 2x - \dfrac{1}{4}\ln(1 + 4x^2)$

55. $y = 8\arcsin \dfrac{x}{4} - \dfrac{x\sqrt{16 - x^2}}{2}$

56. $y = 25\arcsin \dfrac{x}{5} - x\sqrt{25 - x^2}$

57. $y = \arctan x + \dfrac{x}{1 + x^2}$

58. $y = \arctan \dfrac{x}{2} - \dfrac{1}{2(x^2 + 4)}$

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Finding an Equation of a Tangent Line **In Exercises 59–64, find an equation of the tangent line to the graph of the function at the given point.**

59. $y = 2 \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{3}\right)$

60. $y = \frac{1}{2} \arccos x, \quad \left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$

61. $y = \arctan \frac{x}{2}, \quad \left(2, \frac{\pi}{4}\right)$

62. $y = \operatorname{arcsec} 4x, \quad \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

63. $y = 4x \arccos(x - 1), \quad (1, 2\pi)$

64. $y = 3x \arcsin x, \quad \left(\frac{1}{2}, \frac{\pi}{4}\right)$

Linear and Quadratic Approximations **In Exercises 65–68, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2} f''(a)(x - a)^2$$

of the function f at $x = a$. Sketch the graph of the function and its linear and quadratic approximations.

65. $f(x) = \arctan x, \quad a = 0$

66. $f(x) = \arccos x, \quad a = 0$

67. $f(x) = \arcsin x, \quad a = \frac{1}{2}$

68. $f(x) = \arctan x, \quad a = 1$

Finding Relative Extrema **In Exercises 69–72, find any relative extrema of the function.**

69. $f(x) = \operatorname{arcsec} x - x$

70. $f(x) = \arcsin x - 2x$

71. $f(x) = \arctan x - \arctan(x - 4)$

72. $h(x) = \arcsin x - 2 \arctan x$

Analyzing an Inverse Trigonometric Graph **In Exercises 73–76, analyze and sketch a graph of the function. Identify any relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.**

73. $f(x) = \arcsin(x - 1)$

74. $f(x) = \arctan x + \frac{\pi}{2}$

75. $f(x) = \operatorname{arcsec} 2x$

76. $f(x) = \arccos \frac{x}{4}$

Implicit Differentiation **In Exercises 77–80, use implicit differentiation to find an equation of the tangent line to the graph of the equation at the given point.**

77. $x^2 + x \arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$

78. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

79. $\arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

80. $\arctan(x + y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$

WRITING ABOUT CONCEPTS

81. Restricted Domains Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

82. Inverse Trigonometric Functions Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

83. Finding Values

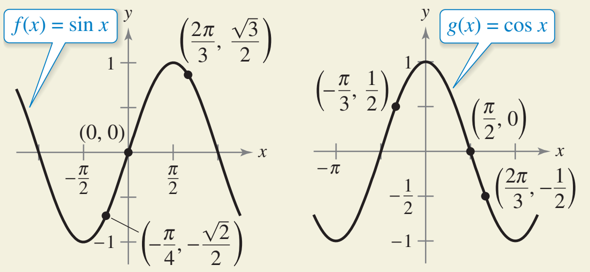
(a) Use a graphing utility to evaluate $\arcsin(\arcsin 0.5)$ and $\arcsin(\arcsin 1)$.

(b) Let

$$f(x) = \arcsin(\arcsin x).$$

Find the values of x in the interval $-1 \le x \le 1$ such that $f(x)$ is a real number.

84. HOW DO YOU SEE IT? The graphs of $f(x) = \sin x$ and $g(x) = \cos x$ are shown below.



(a) Explain whether the points

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\pi}{4}\right), \quad (0, 0) \quad \text{and} \quad \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$$

lie on the graph of $y = \arcsin x$.

(b) Explain whether the points

$$\left(-\frac{1}{2}, \frac{2\pi}{3}\right), \quad \left(0, \frac{\pi}{2}\right), \quad \text{and} \quad \left(\frac{1}{2}, -\frac{\pi}{3}\right)$$

lie on the graph of $y = \arcos x$.

True or False? **In Exercises 85–90, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. Because $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$, it follows that $\arccos \frac{1}{2} = -\frac{\pi}{3}$.

86. $\arcsin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

87. The slope of the graph of the inverse tangent function is positive for all x.

88. The range of $y = \arcsin x$ is $[0, \pi]$.

89. $\frac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

90. $\arcsin^2 x + \arccos^2 x = 1$

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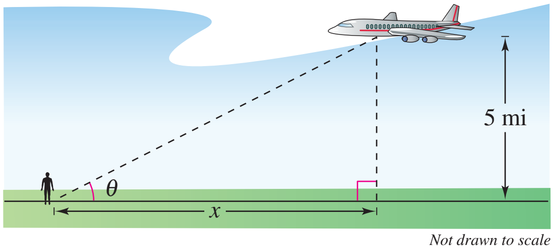
91. Angular Rate of Change An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.

92. Writing Repeat Exercise 91 for an altitude of 3 miles and describe how the altitude affects the rate of change of θ.

93. Angular Rate of Change In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).

(a) Find the position function that yields the height of the object at time t, assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

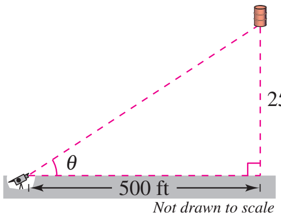


Figure for 93

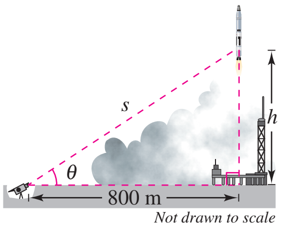


Figure for 94

94. Angular Rate of Change A television camera at ground level is filming the lift-off of a rocket at a point 800 meters from the launch pad. Let θ be the angle of elevation of the rocket and let s be the distance between the camera and the rocket (see figure). Write θ as a function of s for the period of time when the rocket is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

95. Maximizing an Angle A billboard 85 feet wide is perpendicular to a straight road and is 40 feet from the road (see figure). Find the point on the road at which the angle θ subtended by the billboard is a maximum.

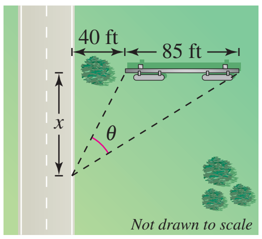


Figure for 95

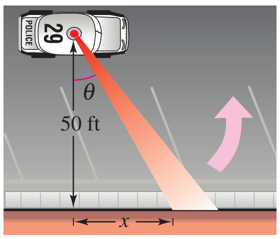


Figure for 96

96. Angular Speed A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of 30 revolutions per minute. Write θ as a function of x. How fast is the light beam moving along the wall when the beam makes an angle of $\theta = 45^\circ$ with the line perpendicular from the light to the wall?

97. Proof

(a) Prove that $\arctan x + \arctan y = \arctan \dfrac{x + y}{1 - xy}$, $xy \neq 1$.

(b) Use the formula in part (a) to show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}.$$

98. Proof Prove each differentiation formula.

(a) $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$

(b) $\dfrac{d}{dx}[\text{arccot } u] = \dfrac{-u'}{1 + u^2}$

(c) $\dfrac{d}{dx}[\text{arcsec } u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$

(d) $\dfrac{d}{dx}[\text{arccsc } u] = \dfrac{-u'}{|u|\sqrt{u^2 - 1}}$

99. Describing a Graph

(a) Graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$.

(b) Describe the graph of f.

(c) Verify the result of part (b) analytically.

100. Think About It Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Why isn't the graph of g the line $y = x$?

(b) Determine the extrema of g.

101. Maximizing an Angle In the figure, find the value of c in the interval $[0, 4]$ on the x-axis that maximizes angle θ.

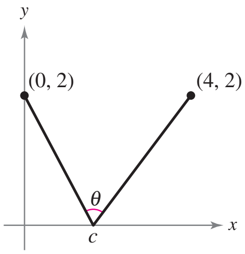


Figure for 101

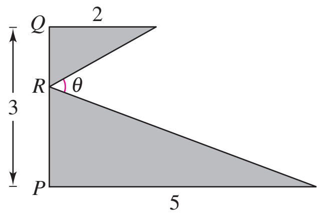


Figure for 102

102. Finding a Distance In the figure, find PR such that $0 \leq PR \leq 3$ and $m\angle\theta$ is a maximum.

103. Proof Prove that $\arcsin x = \arctan\left(\dfrac{x}{\sqrt{1 - x^2}}\right)$, $|x| < 1$.

104. Inverse Secant Function Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup [\pi, 3\pi/2)$.

(a) Sketch the graph $y = \text{arcsec } x$ using this range.

(b) Show that $y' = \dfrac{1}{x\sqrt{x^2 - 1}}$.

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